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by Ye. M. Lifshits

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ELECTRON GAS IN A MAGNETIC FIELD

Ye. M. Lifshits

The kinetic equation is derived for a gas consisting of charged particles in a magnetic field. This equation is used to determine the relaxation time for establishing Maxwell distribution in this gas as well as its thermoconductivity. A bundle of charged particles -- plane and cylindrical -- is investigated. The dependence of the bundle width on time is determined by means of this equation. /390*

This article investigates certain static properties of a gas consisting of charged particles in a magnetic field. For this purpose, the kinetic equation is derived which determines the distribution function of the particles in this gas. This equation may be employed to determine the relaxation time for establishing static equilibrium with respect to the particle velocity, i.e., the time required to establish Maxwell equilibrium [formulas (18) and (19)]. The thermoconductivity of this gas is also determined [formula (28)]. A bundle of charged particles directed along the magnetic field is studied. This bundle will be expanded under the influence of Coulomb interaction between the particles. The dependence of the bundle width on time is determined [formulas (31) and (32)].

1. Kinetic Equation

A study performed by Landau[(1)] derived the general kinetic equation for a gas, whose particles interact with each other according to the Coulomb law. Let each particle be described by the quantities p_i, $i = 1, 2$ (the components of the particle impulse were described by these quantities in I). $n(p_i)$ is the distribution function of the particles with respect to the values of p_i. When two particles collide, the quantities p_i and p_i' which determine them change, respectively, into $p_i + \Delta p_i$ and $p_i' + \Delta p_i'$. During the Coulomb interaction of particles, only those collisions between particles are important for the kinetic equation in which p_i and p_i' change very little -- i.e., $\Delta p_i \ll p_i$, $\Delta p_i' \ll p_i'$ (see I).

Let dW by the probability (per unit of time) for the collision of a particle with the values p_i with a particle having values p_i' during which p_i and p_i' change into $p_i + \Delta p_i$ and $p_i' + \Delta p_i'$. The product $dWn(p_i)n(p_i')$ is the number of such collisions (per one second). dW may be written in the form $dW = w d\tau' d\tau_\Delta$,

* Numbers in the margin indicate pagination in the original foreign text.

(1) L. Landau, Sow. Phys. 10, 154, 1936. This is designated as I below.

where w is, generally speaking, the function of p_i, p_i', Δp_i, $\Delta p_i'$, and $d\tau' = dp_1'dp_2'...d\tau_\Delta$ is the product of the differentials of the parameters determining the collision.

The kinetic equation is then

$$\frac{\partial n}{\partial t}+\frac{\partial j_i}{\partial p_i}=0 \tag{1}$$

(summation is everywhere indicated by a sign which is repeated twice), /391
where j_i is the flux components in p_i - space, which equal the following according to I

$$j_i=-\int d\tau' d\tau_\Delta w\left\{\frac{\Delta p_i\Delta p_k}{2}n'\frac{\partial n}{\partial p_k}+\frac{\Delta p_i\Delta p_k'}{2}n\frac{\partial n'}{\partial p_k'}\right\} \tag{2}$$

[we may write $n' = n(p_i')$].

Let us now turn to a gas consisting of charged particles, in which we are interested (with the charges e and the masses m) in a uniform magnetic field H. We shall select the direction of the field H as the z axis. If we abstract from collisions between particles, the motion of each of the particles in this field may be determined by the equations of motion

$$\ddot{x}-\omega\dot{y}=0,\quad \ddot{y}+\omega\dot{x}=0,\quad \ddot{z}=0, \tag{3}$$

where

$$\omega=\frac{eH}{mc} \tag{4}$$

(c - speed of light). The solution of these equations is

$$x=X+r\cos(\omega t+\alpha),\quad y=Y-r\sin(\omega t+\alpha);\quad z=v_z t+\text{const.} \tag{5}$$

α is the initial phase which depends on the selection of the origin of time, and X, Y, v_z, r are constants. Thus, each particle moves along a spiral line with the radius r and with the axis parallel to the z axis. The quantities X, Y are the coordinates of the spiral line axis. The particle velocity along the direction of the field is v_z. The particle velocity in a direction perpendicular to the field, which we may designate by v, equals

$$v=r\omega. \tag{6}$$

The total particle velocity if $\sqrt{v^2+v_z^2}$.

We may select the terms X, Y, v_z, v as quantities describing a particle. The distribution function is n(X, Y, v_z, v). We must set $p_1 = v$, $p_2 = v_z$, $p_3 = X$, $p_4 = Y$ in (2). However, the kinetic equation itself (1) must now be written in a somewhat different manner. When (1) and (2) were derived, it was assumed that a volume element of p_i-space equals $d\tau = dp_1\, dp_2...$ The elements v, v_2, X, of Y-space equals $d\tau = v\,d\,v\,d\,v_z\,dX\,dY$, since v and v_z

are the cylindrical coordinates for velocity.

Therefore, in order to directly transpose (1) and (2) to our case, we would have to select the variables X, Y, v_z, $v^2/2$ for which $d\tau = dXdYdv_z dv^2/2$. Taking the fact into account that

$$\Delta \frac{v^2}{2} = v\Delta v, \quad \frac{\partial}{\partial \frac{v^2}{2}} = \frac{1}{v} \frac{\partial}{\partial v},$$

we may readily see that equation (1) may be expressed as follows in the case of the variables X, Y, v, v_z:

$$\frac{\partial n}{\partial t} + \frac{\partial j_x}{\partial X} + \frac{\partial j_y}{\partial Y} + \frac{\partial j_{v_z}}{\partial v_z} + \frac{1}{v} \frac{\partial v j_v}{\partial v} = 0, \tag{7}$$

where the flux components are determined by equations (2), just as previously.

We must now determine the changes Δv, Δv_z, ΔX, ΔY and $\Delta v'$, ... of the /392 quantities v, v_z, X, Y and v', ... when two particles collide. The equations of motion for a particle, with allowance for its interaction with another particle, are

$$\ddot{x} - \omega\dot{y} = -\frac{1}{m}\frac{\partial U}{dx}, \quad \ddot{y} + \omega\dot{x} = -\frac{1}{m}\frac{\partial U}{\partial y}, \quad \ddot{z} = -\frac{1}{m}\frac{\partial U}{\partial z}, \tag{8}$$

where U is the potential energy of interaction. Let us solve them by successive approximations. In the zero approximation, we have equation (3) with solution (5). In the first approximation, we must substitute the unperturbed solution in the right hand side of equation (8) -- i.e., (5) -- and a similar solution for the second particle.

We have the following from the third equation of (8)

$$v_z = -\frac{1}{m}\int \frac{\partial U}{\partial z} dt;$$

The total change in v_z during a collision is

$$\Delta v_z = -\frac{1}{m}\int_{-\infty}^{+\infty} \frac{\partial U}{\partial z} dt. \tag{9}$$

The solution of the first two equations may be written in the form (5), where X, Y and v are the variables, however. In order to find ΔX and ΔY, let us rewrite (5) in the following form

$$x = X - \frac{\dot{y}}{\omega}, \quad y = Y + \frac{\dot{x}}{\omega}.$$

Differentiating once with respect to time we have

$$\omega\dot{X} = \ddot{y} + \omega\dot{x}, \quad \omega\dot{Y} = \omega\dot{y} - \ddot{x} \tag{10}$$

and substituting (8), we obtain

$$\dot{X}=-\frac{1}{m\omega}\frac{\partial U}{\partial y},\quad \dot{Y}=\frac{1}{m\omega}\frac{\partial U}{\partial x}.$$

Thus the changes are:

$$\Delta X=-\frac{1}{m\omega}\int_{-\infty}^{+\infty}\frac{\partial U}{\partial y}dt,\quad \Delta Y=\frac{1}{m\omega}\int_{-\infty}^{+\infty}\frac{\partial U}{\partial x}dt. \tag{11}$$

Finally, in order to find Δv we have

$$v^2=\dot{x}^2+\dot{y}^2,$$

from which we have

$$v\dot{v}=\dot{x}\ddot{x}+\dot{y}\ddot{y},$$

or, substituting $\dot{x}$ and $\dot{y}$ from (10), we have

$$v\dot{v}=\dot{X}\ddot{x}+\dot{Y}\ddot{y}$$

In the desired first approximation for $\dot{v}$, we must substitute $v = r\omega$, and $\ddot{x}$ and $\ddot{y}$ from the zero approximation (5), i.e., /393

$$\ddot{x}=-r\omega^2\cos(\omega t+\alpha),\quad \ddot{y}=r\omega^2\sin(\omega t+\alpha).$$

Utilizing the results obtained as well,

$$m\omega\dot{X}=-\frac{\partial U}{\partial y},\quad m\omega\dot{Y}=\frac{\partial U}{\partial x},$$

we obtain

$$\dot{v}=\frac{1}{m}\frac{\partial U}{\partial x}\sin(\omega t+\alpha)+\frac{1}{m}\frac{\partial U}{\partial y}\cos(\omega t+\alpha),$$

from which we have

$$\Delta v=\frac{1}{m}\int_{-\infty}^{+\infty}\left\{\frac{\partial U}{\partial x}\sin(\omega t+\alpha)+\frac{\partial U}{\partial y}\cos(\omega t+\alpha)\right\}dt. \tag{12}$$

The potential energy of the Coulomb interaction of both colliding particles is

$$U=\frac{e^2}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}},$$

or

$$U=e^2\left\{\left[(X-X')+\frac{v}{\omega}\cos(\omega t+\alpha)-\frac{v'}{\omega}\cos(\omega t+\alpha')\right]^2+\right.$$
$$\left.+\left[(Y-Y')-\frac{v}{\omega}\sin(\omega t+\alpha)+\frac{v'}{\omega}\sin(\omega t+\alpha')\right]^2+(v_z-v_z')^2t^2\right\}^{1/2}. \tag{13}$$

We have thus substituted the unperturbed trajectory of motion (5) and the corresponding motion for another particle. The origin of time is taken as the moment when the particles pass by each other; α- and α'- are the phases at the time of collision.

Substituting (13) in (9), (11) and (12), making the substitution $\omega t = u$, we obtain

$$\Delta X = \frac{e^2}{m}\int_{-\infty}^{+\infty} [(Y-Y')\omega - v\sin(u+\alpha) + v'\sin(u+\alpha')]\,du / \{[(X-X')\omega + v\cos(u+\alpha) - v'\cos(u+\alpha')]^2 + [(Y-Y')\omega - v\sin(u+\alpha) + v'\sin(u+\alpha')]^2 + (v_z - v_z')^2 u^2\}^{3/2}$$

$$\Delta Y = \frac{e^2}{m}\int_{-\infty}^{+\infty} - \frac{[(X-X')\omega + v\cos(u+\alpha) - v'\cos(u+\alpha')]\,du}{\{\ldots\}^{3/2}}, \quad (14)$$

$$\Delta v = \frac{e^2\omega}{m} \times \int_{-\infty}^{+\infty} \frac{[-(X-X')\omega\sin(u+\alpha) - (Y-Y')\omega\cos(u+\alpha) + v'\sin(\alpha-\alpha')]\,du}{\{\ldots\}^{3/2}} \qquad (14)$$

$$\Delta v_z = \frac{e^2\omega}{m}\int_{-\infty}^{+\infty} \frac{u(v_z - v_z')\,du}{\{\ldots\}^{3/2}}.$$

The quantities $\Delta X'$, $\Delta Y'$, $\Delta v_z'$, $\Delta v'$ are obtained directly by replacing /394 the primed quantities by the non-primed quantities, and vice versa. We apparently have

$$\Delta X' = -\Delta X,\ \Delta Y' = -\Delta Y,\ \Delta v_z' = -\Delta v_z. \qquad (15)$$

The phases α and α' are the parameters determining the collision. Integration in (2) must be performed over them, where we must write

$$d\tau_\Delta = d\alpha\, d\alpha'.$$

In view of the conservation of energy during collisions, we have

$$v\Delta v + v'\Delta v' + v_z'\Delta v_z' + v_z\Delta v_z = 0.$$

Keeping this in mind, we may readily see that the Maxwell distribution satisfies equation (1), namely: each of the flux components j_i vanishes (2).

In one second, a particle v, v_z, X, Y apparently undergoes $|v_z - v_z'|n'd\tau'$ $d\tau_\Delta$ collisions with the particles v', v_z', X', Y', which are accompanied by a change in their coordinates and velocities for specific Δv_z, ... Consequently, we have

$$w = |v_z - v_z'|. \qquad (16)$$

In connection with equations (1) and (2), we must make the following statement. When these equations were derived in I, the authors made use of the fact that the probability of collision with the transition $p_i \to p_i + \Delta p_i$, $p_i' \to p_i' + \Delta p_i'$ equals the probability of the inverse transition $p_i + \Delta p_i \to p_i$, $p_i' + \Delta p_i' \to p_i'$. However, this changes somewhat in a magnetic field, due to the fact that when the sign of time changes, the sign for the magnetic field must change. Therefore, the probability of the direct process equals the probability of the inverse process in the opposite field direction. Due to this fact, when equations are derived for j_i, generally speaking, terms of the first order in Δp_i do not vanish (see I). However, it may be shown that these terms all vanish in the cases considered below.

2. Relaxation Time

Let us examine a gas composed of charged particles in a magnetic field. We shall assume that the linear dimensions of the gas D are large as compared with the mean radius of the spiral line of the particle thermal motion, i.e.,

$$D \gg \frac{v_0}{\omega} \sim \frac{c\sqrt{mkT}}{eH}. \tag{17}$$

(k - Boltzmann constant, T - temperature). We shall employ the term v_o to designate the mean thermal velocity of a particle: $v_o \sim \sqrt{kT|m}$.

Let us determine the relaxation time, i.e., the time required to establish Maxwell distribution of the gas under consideration (distribution of particles over the coordinates, i.e., gas density, is thus uniform). The desired time may be determined by those terms of the kinetic equation which contain j_v and j_{v_z}.

The relaxation times for establishing equilibrium with respect to the velocities v and v_z -- i.e., perpendicularly and parallel to the field -- are different.

Let us determine the relaxation time for equilibrium with respect to the velocity v. For this purpose, we must determine the order of magnitude of the term $\frac{1}{v}\frac{\partial}{\partial v} v j_v$ in (7). There are three terms in j_v with the products /395 $\Delta v \Delta_{\chi 1}$ $\Delta v \Delta v_z$ $(\Delta v)^2$. An examination shows that the first of these terms is exponentially small (as $e^{-D\omega/v_0}$), and the second is small as compared with the third logarithmically (by a factor of $\ln \frac{mv_0^3}{e^2\omega}$). Therefore, only the term with $(\Delta v)^2$ is important.

An investigation shows that in the case of $d \gg v_0/\omega$ [$d = \sqrt{(X - X')^2 + (Y - Y')^2}$ is the distance between colliding particles] Δv is exponentially small. In the case of $d \sim v_0/\omega$ (i.e., for $d \sim v/\omega$), as may be seen from (14), we have

$$\Delta v \sim \frac{e^2\omega}{mv\,|v_z - v_z'|}.$$

When this expression is substituted in j_v, the integral with respect to $|v_z - v_z'|$ diverges logarithmically. This occurs due to the fact that for small $|v_z - v_z'|$ Δv is large, and the formulas derived lose their applicability. We may take the following as the lower integration limit with respect to $|v_z - v_z'|$

$$|v_z - v_z'| \sim \frac{e^2\omega}{mv^2} \sim \frac{e^2\omega}{kT},$$

and v_0 as the upper integration limit.

When determining j_v, we must keep the fact in mind that $v \sim \sqrt{kT|m}$, and integration over dX' and dY' is performed in the region v/ω. We then have

$$\frac{\partial j_v}{\partial v} \sim \frac{e^4 n\nu}{m^{1/2}(kT)^{3/2}} \ln \frac{m^{1/2}(kT)^{3/2}c}{e^3 H},$$

where ν is the gas density, i.e., the number of particles per unit of volume. According to the kinetic equation, we must equate $\frac{\partial j_v}{\partial v}$ to $\frac{\partial n}{\partial t}$. The derivative $\frac{\partial n}{\partial t} \sim \frac{n}{t_v}$ where t_v is the desired relaxation time. We thus find

$$t_v \sim \frac{m^{1/2}(kT)^{3/2}}{e^4\nu \ln \frac{m^{1/2}(kT)^{3/2}c}{e^3 H}}. \tag{18}$$

Let us now determine the relaxation required to establish equilibrium with respect to the velocities v_z along the field. An examination of the integral determining Δv_z [see (14)] shows that Δv_z is exponentially small in the case of $d \gg \frac{v_0}{\omega}$ (as $e^{-D\omega/v_0}$) and also for $|v_z - v_z'|$ (as $e^{-v_0/|v_z - v_z'|}$). In the case of $d\omega \sim v_0$ and $|v_z - v_z'| \sim v_0$, we have

$$\Delta v_z \cong \frac{e^2\omega}{mv_0^2} \cong \frac{e^2\omega}{kT}.$$

Similarily to (18), we find the following by means of this expression

$$t_{vz} \cong \frac{m^{1/2}(kT)^{3/2}}{e^4\nu}. \tag{19}$$

It may be seen from a comparison of (18) and (19) that equilibrium with respect to the velocities v perpendicular to the field is established more rapidly than with respect to the velocity v_z parallel to the field.

Substituting numerical values, we have /396

$$t_v \cong \frac{T^{3/2}}{\nu \ln 10 \frac{T^{3/2}}{H}},$$

$$t_{v_z} \cong \frac{T^{3/2}}{\nu}.$$

In the case of $H \cong 1000$ gauss, $T \cong 10{,}000°$, $\nu \cong 10^{8}\ \mathrm{cm}^{-3}$

$$t_v \sim 10^{-3}\ \mathrm{sec.},\ t_{v_z} \sim 10^{2}\ \mathrm{sec.}$$

It is difficult to compare these quantities with experimental data, since there are no accurate data pertaining to a purely electron (without ions) gas in a magnetic field.

3. Thermal Conductivity

We shall assume that in each small volume of the gas to be studied Maxwell distribution has already been established. However, only the gas density and its temperature are different in different locations. We shall thus assume that the gas is non-uniform in one direction, which we shall select as the x axis. The density and temperature are functions of the coordinate X and the time t. Thus, we may write n in the following form

$$n = 2\pi\left(\frac{m}{2\pi kT}\right)^{3/2} \nu e^{-\frac{m(v^2+v_z^2)}{2kT}}, \tag{20}$$

where $T = T(X,t)$ and $\nu = \nu(X,t)$ (ν is the number of particles per 1 cm^3).

In order to find the equations determining ν and T, we shall proceed as follows. We shall integrate (7) from both sides over $v\,dv\,dv_z\,dY$ with respect to all possible values. It is trivial to integrate over dY, since nothing depends on Y, and it yields simply the gas dimensions along the y axis which are reduced in both directions (7). The terms $\frac{1}{v}\frac{\partial v j_v}{\partial v}$ and $\frac{\partial j_{v_z}}{\partial v_z}$ of the integration thus vanish (since $j_v = j_{v_z} = 0$ at the limits), and we find:

$$\frac{\partial}{\partial t}\int n\,d\tau_v = -\frac{\partial}{\partial X}\int j_x\,d\tau_v,\quad d\tau_v = v\,dv\,dv_z,$$

or, substituting (20) in the left hand part, we obtain

$$-\frac{\partial\nu}{\partial t} = \frac{\partial}{\partial X}\int j_x\,d\tau_v. \tag{21}$$

Multiplying (7) from both sides by the energy ε of the particle $\left[\varepsilon = \frac{m}{2}(v^2 + v_z^2)\right]$ and integrating over $d\tau_v$, we obtain

$$-\frac{\partial}{\partial t}\int n\varepsilon\,d\tau_v = \frac{\partial}{\partial X}\int j_x\varepsilon\,d\tau_v - \int m v_z j_{v_z} d\tau_v - \int m v j_v d\tau_v,$$

or, substituting in the left hand side of (20), we obtain

$$-\frac{3}{2}\frac{\partial(\nu T)}{\partial t} = \frac{\partial}{\partial X}\int j_x\varepsilon\,d\tau_v - \int m v_z j_{v_z}\,d\tau_v - \int m v j_v\,d\tau_v. \tag{22}$$

We may substitute the expression for j_x from (2) in (21). As may /397 be seen from (2), j_x consists of three terms (corresponding to $k = 1, 2, 3$), containing $\Delta X\Delta v$, $\Delta X\Delta v_z$, $(\Delta X)^2$, respectively. An examination shows that the first two are exponentially small. Collisions at large distances ($d \gg v_0\omega$) play a role in the third term, which is verified by the final expression for j_x. In the case of $d \gg v_0\omega$ the first one of formulas (14) yields

$$\Delta X = \frac{e^2}{m}\int_{-\infty}^{+\infty} \frac{(Y - Y')\omega\, du}{[d^2\omega^2 + (v_z - v_z')^2u^2]^{3/2}} = \frac{2e^2(Y - Y')}{d^2\omega m\,|v_z - v_z'|}. \tag{23}$$

We obtain the following from (12) for j_x [assuming that $\omega = |v_z - v_z'|$ -- see (16) -- and that $\Delta X' = -\Delta X$]:

$$j_x = \int |v_z - v_z'| \frac{(\Delta X)^2}{2}\left\{n\frac{\partial n'}{\partial X'} - n'\frac{\partial n}{\partial X}\right\} d\tau' d\alpha\, d\alpha'.$$

Integration over $d\alpha\, d\alpha'$ yields $4\pi^2$. In addition, let us substitute (23) and let us integrate over dY'. Since the gas dimensions are large, and the integral with respect to dY' converges, let us integrate from $-\infty$ to $+\infty$. We obtain:

$$j_x = \frac{4\pi^3e^4}{m^2\omega^2}\int \frac{dX' d\tau_v'}{|X - X'|\,|v_z - v_z'|}\left(n\frac{\partial n'}{\partial X'} - n'\frac{\partial n}{\partial X}\right). \tag{24}$$

The integral with respect to dv_z' diverges logarithmically for small $|v_z - v_z'|$. This is related to the fact, which was already mentioned, that for small $|v_z - v_z'|$ ΔX is large, and the formulas derived are not applicable. Therefore, the lower integration limit with respect to $|v_z - v_z'|$ may be selected at the point where $\Delta X \gtrsim X$, i.e., where

$$|v_z - v_z'| \cong \frac{e^2}{m\omega D^2}.$$

When substituting the expression (20) for n in (24), we encounter the following integral, for example

$$\int \frac{e^{-\frac{mv_z'^2}{2kT'}}}{|v_z - v_z'|}\, dv_z'$$

$[T' = T(X')]$. The integrand is large for v_z which is close to v_z', and rapidly decreases after $v_z' \sim \sqrt{\frac{kT}{m}}$. Therefore, we have

$$\int \frac{e^{-\frac{mv_z'^2}{2kT'}}}{|v_z - v_z'|}\, dv_z' \cong 2e^{-\frac{mv_z^2}{2kT'}} \int_{\frac{e^2}{m\omega D^2}}^{v_0} \frac{d(v_z' - v_z)}{(v_z' - v_z)} = 2e^{-\frac{mv_z^2}{2kT'}} \ln\frac{v_0\omega D^2 m}{e^2}.$$

Finally, substituting the expression (20) for n in (24) and substituting j_x in (21), after the computation we obtain

$$-\frac{\partial \nu}{\partial t} = \frac{2^{3/2}\pi^{5/2}e^4}{m^{3/2}\omega^2}\ln\frac{(mkT)^{1/2}\omega D^2}{e^2}\frac{\partial}{\partial X}\int \frac{dX'}{|X - X'|\sqrt{kT + kT'}} \times \\ \times\left\{2\left(\nu\frac{\partial \nu'}{\partial X'} - \nu'\frac{\partial \nu}{\partial X}\right] - \frac{\nu\nu'}{T + T'}\left(\frac{\partial T'}{\partial X'} - \frac{\partial T}{\partial X}\right)\right\}. \tag{25}$$

The same procedure may be followed for equation (22). It is thus found that the third term to the right is logarithmically small as compared with the first (by a factor of ln $\omega D/v_0$), and the second is smaller by a factor /398 of ln $(mkT)^{\frac{1}{2}}\omega D^2/e^2$. We shall disregard both of these terms. Utilizing the expressions which we obtained for $\nu/\partial t$, after the calculation we obtain

$$-\nu\frac{\partial T}{\partial t}=$$

$$=\frac{2^{3/2}\pi^{5/2}e^4}{m^{3/2}\omega^2k^{1/2}}\ln\frac{(mkT)^{1/2}\omega D^2}{e^2}\frac{\partial}{\partial X}\int\frac{dX'}{|X-X'|(T+T')^{3/2}}\left\{-\frac{2}{3}\left(\nu\frac{\partial\nu'}{\partial X'}-\nu'\frac{\partial\nu}{\partial X}\right)T^2+\right. \quad (26)$$

$$\left.+\frac{\nu\nu'}{3(T+T')}\left[\frac{\partial T'}{\partial X'}3T^2-\frac{\partial T}{\partial X}(5T^2+6T'^2+8TT')\right]\right\}.$$

Let us expand the integrand with respect to (X - X'). In the first approximation, this may be reduced to the fact that we assume $T = T'$, $\frac{\partial T'}{\partial X'} = \frac{\partial T}{\partial X}$, $\nu = \nu'$, etc. The first term (with the derivatives of density) in this approximation vanishes, i.e., it is less than the second (with the derivatives of temperature). In addition, the following integral remains

$$\int\frac{dX'}{|X-X'|}=2\int_{v_0/\omega}^{D}\frac{d(X-X')}{(X-X')}=2\ln\frac{D\omega}{v_0}.$$

As a result, we obtain

$$\frac{\partial T}{\partial t}=\frac{1}{\nu}\frac{\partial}{\partial X}\left\{\frac{16\pi^{5/2}e^4\nu^2}{3m^{3/2}\omega^2}\cdot\ln\frac{D\omega}{v_0}\ln\frac{\omega D^2mv_0}{e^2}\frac{1}{\sqrt{kT}}\frac{\partial T}{\partial X}\right\}. \quad (27)$$

This is the customary equation of thermal conductivity

$$c_v\frac{\partial T}{\partial t}=\frac{\partial}{\partial X}\left(\vartheta\frac{\partial T}{\partial X}\right),$$

where ϑ is the thermal conductivity coefficient, and $c_v = \frac{3}{2}\nu k$ is the heat capacity (for a constant volume). We have the following expression for the thermal conductivity coefficient

$$\vartheta=\frac{8\pi^{5/2}e^2\nu^2c^2m^{1/2}k^{1/2}}{H^2\sqrt{T}}\ln\frac{eHD}{e\sqrt{mkT}}\ln\frac{HD^2(kT)^{1/2}}{cem^{1/2}} \quad (28)$$

(we have substituted $\omega = \frac{eH}{mc}$). Numerically, we have

$$\vartheta=10^{-17}\frac{\nu^2}{H^2\sqrt{T}}\ln 50\frac{HD}{T^{1/2}}\ln 3\cdot 10^4HD^2T^{1/2}\ \frac{\text{erg.}}{^\circ\text{C cm}\cdot\text{sec.}}.$$

H is given in gauss, T - in degrees, D - in cm, and ν is given in cm^{-3}.

Let us compare the quantities $\frac{1}{\nu}\frac{\partial\nu}{\partial t}$ and $\frac{1}{T}\frac{\partial T}{\partial t}$. We saw that in equation (26) for $\frac{\partial T}{\partial t}$ the term with the derivatives of temperature was $\frac{D\omega}{v_0}$ times greater than the terms with the derivatives of density, which were therefore disregarded. In equation (25) for $\frac{\partial\nu}{\partial t}$, all the terms vanished in the case of

$\nu = \nu'$, $T = T'$, etc. Thus, $\frac{1}{\nu}\frac{\partial \nu}{\partial t}$ is $\ln \frac{D\omega}{v_0}$ times smaller than $\frac{1}{T}\frac{\partial T}{\partial t}$. /399

Consequently, we arrive at the result that the temperature is equalized in a gas more rapidly than the density.

4. Bundle of Charged Particles

Let us now investigate a bundle of charged particles which is directed along a magnetic field (this direction is again selected as the z axis). In each cross section, the bundle density depends on the X and Y coordinates. We shall employ D to designate the order of magnitude of the bundle thickness. It is assumed that the inequality (17) is satisfied for D, just as previously.

The bundle is non-uniform along the z axis. However, instead of examining a bundle which is non-uniform over all three directions, we may examine a certain section of the bundle (along its length) in a coordinate system which moves along with it -- i.e., with a velocity equalling the bundle velocity. In this system, the particles have only thermal motion. We then have a gas which is uniform along the z axis, and is not uniform along the x and y axes, whose density depends on time, however. We shall proceed as follows.

At the end of section three, we found that the temperature is equalized more rapidly than the density. Therefore, we shall assume that the bundle temperature has already been equalized, i.e., T = const. Let us write the equation for $\frac{\partial \nu}{\partial t}$. It will now differ somewhat from (25), due to the fact that the bundle is not uniform along both the x and y axes. In this connection, the term $\frac{\partial j_y}{\partial Y}$ also remains in the kinetic equation (7), and terms with derivatives with respect to Y remain in the fluxes j_x and j_y. The expression for ΔY differs from ΔX (23) only in the fact that (X - X') appear, instead of (Y - Y') [see (14)]. As a result, we obtain the following equation

$$-\frac{\partial \nu}{\partial t} = \frac{8\pi^{3/2} e^4}{m^{3/2}\omega^2\sqrt{kT}} \ln \frac{(mkT)^{1/2}\omega D^2}{e^2} \left\{ \frac{\partial}{\partial X} \int \frac{dX'\, dY'}{[(X-X')^2+(Y-Y')^2]^2} \times \right. \tag{29}$$

$$\left. (Y-Y')^2\left(\nu\frac{\partial \nu'}{\partial X'} - \nu'\frac{\partial \nu}{\partial X}\right) - (Y-Y')(X-X')\left(\nu\frac{\partial \nu'}{\partial Y'} - \nu'\frac{\partial \nu}{\partial Y}\right)\Big] + \frac{\partial}{\partial Y}\int \cdots \right\}.$$

The second term in the parentheses differs from the first term by the transposition of X and Y. It is naturally impossible to integrate over dY' here, since ν is a function of Y.

In the solution of this equation, one dimensional constant must be contained in it, in addition to the constant which it already contains. The integral $\int \nu \, dX\, dY = N$ is the total number of particles in the bundle pertaining to a unit of its length along its direction (z axis). N has the dimensionality

cm^{-1}. The density ν must have the form $\nu = NF(X, Y, t)$, where F is the function with the dimensionality cm^{-2}. Substituting this in (29), we find /400 that only one dimensional constant is included in the equation which results:

$$\frac{e^4 N}{m^{3/2}\omega^2 \sqrt{T}} \ln \frac{(mkT)^{1/2}\omega D^2}{e^2}.$$

Let us examine a bundle with cylindrical symmetry, i.e., a bundle in which ν is only a function of the distance r up to the bundle axis (and of time). X and Y must be expressed in equation (29) by means of polar coordinates, and integration may be performed over the polar angle. However, we may compile only one dimensionless quantity from the dimensional constant indicated above and the independent variables r and t

$$\frac{t}{r^4}\frac{e^4 N}{m^{3/2}\omega^2\sqrt{kT}} \ln \frac{(mkT)^{1/2}\omega D^2}{e^2}.$$

Consequently, ν must have the following form

$$\nu = \frac{N}{r^2} f\left(\frac{e^4 N}{m^{3/2}\omega^2\sqrt{kT}} \ln \frac{(mkT)^{1/2}\omega D^2}{e^2} \cdot \frac{t}{r^4}\right) \quad (30)$$

(the function F in $\nu = NF$ may be written in the form $F = f/r^2$, where f is the dimensionless function).

Under the influence of Coulomb repulsion between particles, the bundle expands with the passage of time. Let us determine the change in the thickness D of the bundle with time, i.e., the width of the region outside of which the density is very small. It may be seen directly from (30) that the dependence of D on time is determined by the following formula

$$\frac{D^4 - D_0^4}{t} = \mathrm{const} \sim \frac{e^2 N m^{1/2} c^2}{H^2\sqrt{kT}} \ln \frac{(kT)^{1/2} H D^2}{e m^{1/2} c}, \quad (31)$$

where D_0 is the thickness in the case of $t = 0$. Instead of examining a gas with a density which depends on r and t, as was indicated above, we may speak of a bundle which is *directed along the z axis*. Then (31) determines the change in the bundle width along its direction. Instead of t, we must write the coordinate z which is divided by the bundle velocity.

A somewhat different result is obtained for a plane bundle. The bundle is now very wide (theoretically infinite) and uniform along the y axis. In each cross section, the bundle density depends only on X. The bundle is symmetrical with respect to a certain plane (the yz plane). The equation for $\frac{\partial\nu}{\partial t}$ is now obtained from (25), assuming that T = const., or from (29), cancelling the derivatives with respect to Y. The integral $\int \nu dX = N$ is now the total number of particles pertaining to a unit of length along the bundle direction, and along its width over the y axis. N now has the dimensionality $\frac{1}{cm^2}$.

It is now found that the bundle thickness (along the x axis) changes with time according to the following equation, in a manner which is completely similiar to that of a cylindrical bundle:

$$\frac{D^3 - D^3_0}{t} = \text{const} \sim \frac{e^2 N m^{1/2} c^2}{H^2 (kT)^{1/2}} \ln \frac{(kT)^{1/2} D^2 H}{m^{1/2} ce}. \tag{32}$$

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